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On Continuous Selections for Metric Projections in Spaces of Continuous Functions

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X is a compact Hausdorff space and $C(X)$ the Banach space of real-valued continuous functions on X. Amongst other results it is shown that, if M is a closed linear subspace of $C(X)$ such that no nonzero member of M is zero on a nonempty open subset of X and for each f in $C(X)$ the metric projection $P_M(f)$ of f onto M is nonempty and finite-dimensional, then if there is a continuous selection for P_M it is unique. An example is given of a five-dimensional subspace M of $C([-1, 1])$ which is non-Chebyshev and for whose metric projection P_M there is a unique continuous selection. This example shows that a result claimed by other authors in a previous paper on this subject is false.

1. INTRODUCTION

Throughout the paper, X will denote a compact Hausdorff space and $C(X)$ the Banach space, with the uniform norm, of real-valued continuous functions on X. If M is a nonempty subset of $C(X)$ then P_M, or simply P, will denote the metric projection of $C(X)$ onto M. Then P_M is a set-valued mapping on $C(X)$ and, writing

$$d(f, M) = \inf\{\|f - p\| : p \in M\}$$

for the distance of $f \in C(X)$ from M, we have

$$P_M(f) = \{p \in M : \|f - p\| = d(f, M)\}.$$

A continuous mapping s of $C(X)$ into M is said to be a continuous selection for P_M if $s(f) \in P_M(f)$ for all $f \in C(X)$.

In two recent papers, Blatter, Morris and Wulbert [1] and Lazar, Wulbert and Morris [2] have studied several questions concerning continuity properties of metric projections in Banach spaces. The present paper is concerned with those sections of [1] and [2] which discuss the lower semicontinuity of P_M and the existence of continuous selections for P_M when M is a closed linear subspace of $C(X)$ with the property that $P_M(f)$ is finite-dimensional for every $f \in C(X)$. The origin of the paper was the discovery of a flaw in [2]. In fact, the main statement concerning $C(X)$ in that paper [2, Theorem 2.1] is false.

In the discussion which follows, M will always be a closed linear subspace of $C(X)$ with the property that for each $f \in C(X)$ the set $P(f) = P_M(f)$ is nonempty and finite-dimensional. Section 2 is concerned with the sets

$$P^*(f) = \{h \in P(f) : d(h, P(g)) \to 0 \text{ as } g \to f\}.$$

The significance for us of P^* is summarised by the following simple proposition.

PROPOSITION 1.1. (i) *P is lower semicontinuous if and only if $P^*(f) = P(f)$ for all $f \in C(X)$.*

(ii) *If s is a continuous selection for P then $s(f) \in P^*(f)$ for all $f \in C(X)$.*

In some respects, the discussion in Section 2 follows closely and leans heavily on that in [1, 2]. In other respects, it is a simplification and refinement of that in those papers. The proofs here are more elementary, and positive results additional to those in [1, 2] are obtained. The basic result of Section 2 is Lemma 2.6 which can be thought of as an approach to a characterization of the sets $P^*(f)$. This lemma yields the necessity of a condition for the lower semicontinuity of P[1, Theorem 2] and also the initially surprising result that if M has the property that no nonzero member of M is zero on a nonempty open subset of X, then if there does exist a continuous selection for P it is unique (Theorem 2.8).[1]

In Section 3 we establish (Theorem 3.10) the existence of a five-dimensional non-Chebyshev subspace M of $C([-1, 1])$ with the property that there does exist a (unique) continuous selection for P_M. This example provides a counter example to the statements of Theorem 2.1 and Corollary 2.5 of [2]. The construction of the subspace M and the proof that there does exist a continuous selection for P_M is formally independent of Section 2. However, it will be apparent that the information obtained from Section 2 was of con-

[1] Professor Wulbert has pointed out to the author that Lemma 2.6 can also be obtained by modifying the proof of [2, Lemma 2.2].

siderable importance in the construction of the example; in particular, the results of Section 2 identify the continuous selection and also establish a necessary condition (Theorem 2.8(ii)) for the existence of one. In this sense, Section 3 is dependent upon Section 2. The uniqueness of the continuous selection for P_M is a direct consequence of Theorem 2.8.

2. On the Existence of Continuous Selections

The aim in this section is to obtain as much information as possible about the sets $P^*(f)$. The mode of attack is to 'perturb' f to obtain functions g related to f in such a way that $P(g)$ can be described (in terms of $P(f)$) and a lower bound for $d(h, P(g))$ obtained when $h \in P(f)$. The essential tool in obtaining lower bounds is the following lemma. It is a slight extension and reformulation of [1, Lemma 2].

LEMMA 2.1. *Let P be a convex subset of a finite-dimensional subspace N of $C(X)$ and let Z be a nonempty subset of X. If $h \in N$ and $p_0 \in P$ have the property that for each neighbourhood U of Z*

$$p_0(x) - h(x) > 0 \text{ for some } x \in U, \tag{1}$$

then there exist $q \in P$ and $r > 0$ such that, if $p \in P$ and

$$\{x : p(x) \geqslant q(x)\} \text{ is a neighbourhood of the set } Z, \tag{2}$$

then

$$\|p - h\| \geqslant r.$$

If the inequality signs in both (1) *and* (2) *are reversed, then the resulting statement is also true.*

Proof. The case $h = 0$ is essentially Lemma 3 of [1] (but it should be noted that the proof as reproduced in [1] requires a small modification). Now suppose that (1) holds. The first statement of the lemma follows from the case $h = 0$ applied to the set $P - h$ and the function $p_0 - h \in P - h$. To obtain the result with inequalities reversed the first result is applied to the set $-P$ and to $-h \in N$ and $-p_0 \in -P$.

Our results will be obtained as the end product of a series of technical lemmas. The following notations will be used. If $A \subseteq C(X)$, then we write

$$Z(A) = \bigcap\{f^{-1}(0) : f \in A\}$$

If $Y \subseteq X$ and $f \in C(X)$, then we write

$$\|f\|_Y = \sup_{x \in Y} |f(x)| .$$

For each $f \in C(X)$ the set $P(f)$ is a nonempty, finite-dimensional, and convex subset of M. Therefore $P(f)$ has a nonempty relative interior which will be denoted by relint $P(f)$.

If $0 \in P(f)$ then the subset $Z(P(f))$ of X is particularly significant—this fact is one of the themes of [1, 2]. (It should be noted that if $p_0 \in P(f)$, then $0 \in P(f - p_0)$ and that if $p_0 \in \operatorname{relint} P(f)$ then $0 \in \operatorname{relint} P(f - p_0)$.) The significance of $Z(P(f))$ is indicated by the following simple lemma. The result of the lemma is well-known and is contained in elementary proofs of Haar's theorem. The lemma will be used in the proofs of Theorems 2.7 and 2.8 and in Section 3 but it is a natural background for some of the intervening lemmas.

LEMMA 2.2. *Let* $0 \in \operatorname{relint} P(f)$. *Then* $x \in Z(P(f))$ *if* $|f(x)| = \|f\|$.

Proof. From the fact that $0 \in \operatorname{relint} P(f)$ it follows that $\|f\| = d(f, M)$ and that if $p \in P(f)$, then $-\lambda p \in P(f)$ for some $\lambda > 0$. Then

$$|f(x) - p(x)| \leqslant \|f\|,$$

$$|f(x) + \lambda p(x)| \leqslant \|f\|,$$

and

$$|f(x)| = \|f\|$$

are only possible if $p(x) = 0$.

The next lemma relates $P(f')$ to $P(f)$ for certain 'perturbations' f' of f.

LEMMA 2.3. *Let* $0 \in P(f)$ *and* $\|f\| = 1$. *Let* $\Gamma^+ = f^{-1}(1)$ *and* $\Gamma^- = f^{-1}(-1)$. *If* $f' \in C(X)$ *is such that* $\|f'\| = 1$, *and, for some open neighbourhood* U^+ *of* Γ^+ *and some open neighbourhood* U^- *of* Γ^-,

$$f'(x) \geqslant f(x) \textit{ for all } x \in U^+$$

and

$$f'(x) \leqslant f(x) \textit{ for all } x \in U^-,$$

then $0 \in P(f')$ *and, for some* $\mu > 0$, $P(f') \subseteq \mu P(f)$.

Proof. It may be supposed that $f(x) > 0$ for all $x \in U^+$ and $f(x) < 0$ for all $x \in U^-$. Let $A = U^+ \cup U^-$ and $B = X \setminus A$.

Then $\|f\|_B < 1$. Let $0 < \lambda < \frac{1}{2}(1 - \|f\|_B)$. Suppose that $p \in P(f')$. It will be shown that $\lambda p \in P(f)$.

The inequality

$$\|f' - p\| \leqslant \|f'\| = 1 \tag{3}$$

gives

$$f(x) - p(x) \leqslant f'(x) - p(x) \leqslant \|f' - p\| \text{ for all } x \in U^+$$

and

$$f(x) - p(x) \geqslant f'(x) - p(x) \geqslant -\|f' - p\| \text{ for all } x \in U^-.$$

Therefore,

$$f(x) - \lambda p(x) \leqslant (1 - \lambda) + \lambda\|f' - p\| \leqslant 1 \qquad \text{for all} \quad x \in U^+$$

and

$$f(x) - \lambda p(x) \geqslant -(1 - \lambda) - \lambda\|f' - p\| \geqslant -1 \quad \text{for all} \quad x \in U^-. \tag{4}$$

Now $\lambda \| p \| \leqslant 2\lambda < 1$ and so the inequalities

$$f(x) - \lambda p(x) > -\lambda p(x) > -2\lambda \qquad \text{for all} \quad x \in U^+$$

and

$$f(x) - \lambda p(x) < -\lambda p(x) < 2\lambda \qquad \text{for all} \quad x \in U^-,$$

together with (4), give

$$\|f - \lambda p\|_A \leqslant 1.$$

We also have

$$\|f - \lambda p\|_B \leqslant \|f\|_B + \lambda\| p \|_B \leqslant \|f\|_B + 2\lambda < 1,$$

and so

$$\|f - \lambda p\| \leqslant 1. \tag{5}$$

This proves that $\lambda p \in P(f)$. Now if there were strict inequality in (3), then there would be strict inequality in (5), in contradiction to $0 \in P(f)$. This proves that $0 \in P(f')$.

The next lemma is needed to show that, when $0 \in P(f)$, every point of the set $Z(P(f))$ is significant. The construction of the lemma is a refinement of one in [1].

LEMMA 2.4. *Let* $f \in C(X)$, $\|f\| = 1$, *and* $0 \in P(f)$. *If* $w \in Z(P(f))$ *and* $|f(w)| < 1$, *then there exist* $f', f'' \in C(X)$ *with*

(i) $\|f'\| = 1$, $f'(w) = 1$, $P(f') = P(f)$ *and* $f'(x) = f(x)$ *whenever* $|f(x)| = 1$,

(ii) $\|f''\| = 1$, $f''(w) = -1$, $P(f'') = P(f)$ *and* $f''(x) = f(x)$ *whenever* $|f(x)| = 1$.

Proof. It is sufficient to construct f', for the same construction when applied to $-f$ will yield f''.

By our general assumptions, the set $P(f)$ is finite-dimensional and therefore compact. Thus the set $f - P(f)$ is compact and equicontinuous. It follows that there is an open neighbourhood W of w such that

$$|f(x) - q(x)| < \tfrac{1}{2}(|f(w)| + 1)$$

for all $q \in P(f)$ and all $x \in W$.

It follows from the equicontinuity of $P(f)$ that the function $\varphi = \inf P(f)$ is a member of $C(X)$. Also $\varphi(w) = 0$ and $\varphi(x) \leqslant 0$ for all $x \in X$. Furthermore $f(x) \leqslant 1 + \varphi(x)$ for all $x \in X$. The space X is compact and Hausdorff, and so normal, and so there exists $\psi \in C(X)$ with $\psi(x) = 0$ for $x \notin W$, $\psi(w) = 1$ and $0 \leqslant \psi(x) \leqslant 1$ for all $x \in X$. Then f' defined by

$$f'(x) = (1 - \psi(x))f(x) + \psi(x)(1 + \varphi(x))$$

has the properties: $f'(x) = f(x)$ for all $x \notin W$ (and, in particular, if $|f(x)| = 1$), $f'(w) = 1$ and $f(x) \leqslant f'(x) \leqslant 1 + \varphi(x)$ for all $x \in X$. Thus $\|f'\| = 1$ and it must be shown that $P(f') = P(f)$.

First it will be shown that $P(f') \subseteq P(f)$. By Lemma 2.3, $0 \in P(f')$ and $P(f') \subseteq \mu P(f)$ for some $\mu > 0$. Suppose that $q \in P(f') \setminus \{0\}$ and let $\alpha = \sup\{\lambda : \lambda q \in P(f)\}$. Then $\alpha > 0$ and $\alpha q \in P(f)$. If it is shown that $\alpha \geqslant 1$ then it will follow that $q \in P(f)$.

By the choice of W, $\|f - \alpha q\|_W < 1$. Let δ be any real number such that $\delta > 0$ and $\delta \| q \|_W \leqslant 1 - \|f - \alpha q\|_W$ Then $(\alpha + \delta) q \notin P(f)$ and so

$$\|f - (\alpha + \delta) q\| = \max\{\|(f - \alpha q) - \delta q\|_W , \|f - (\alpha + \delta) q\|_{X\setminus W}\} > 1.$$

It now follows by the restriction on δ that

$$\|f' - (\alpha + \delta q)\|_{X\setminus W} = \|f - (\alpha + \delta) q\|_{X\setminus W} > 1.$$

Therefore $(\alpha + \delta) q \notin P(f')$ and so $\alpha + \delta > 1$. This is true for all small $\delta > 0$ and so $\alpha \geqslant 1$. That $P(f') \subseteq P(f)$ is now proved.

Suppose that $q \in P(f)$. Then, for all $x \in X$,

$$-1 \leqslant f(x) - q(x) \leqslant f'(x) - q(x) \leqslant 1 + (\varphi(x) - q(x)) \leqslant 1,$$

where the last inequality follows from the definition of φ. Thus $\| f' - q \| \leqslant 1$ and so $q \in P(f')$. This completes the proof.

The next lemma is also an existence theorem for 'perturbations' of f with certain properties. Like the last lemma it is a development of material in [1].

LEMMA 2.5. *Let $f \in C(X)$ have $0 \in P(f)$, $\| f \| = 1$. Suppose that $q \in P(f)$. Then for each $\epsilon > 0$ there is an $f_\epsilon \in C(X)$ with the following properties;*

(i) $\| f - f_\epsilon \| < \epsilon$,

(ii) $q \in P(f_\epsilon) \subseteq P(f)$,

(iii) *if $p \in P(f_\epsilon)$, then*

$\{x : p(x) \geqslant q(x)\}$ *is a neighbourhood of* $Z^+ = f^{-1}(1) \cap Z(P(f))$

and

$\{x : p(x) \leqslant q(x)\}$ *is a neighbourhood of* $Z^- = f^{-1}(-1) \cap Z(P(f))$.

Proof. The lemma will be proved first in the case $q = 0$. One may suppose that $0 < \epsilon < 1$. The subspace sp $P(f)$ of $C(X)$ spanned by $P(f)$ is finite-dimensional and so $\{p \in \mathrm{sp}\, P(f) : \| p \| \leqslant 2\}$ is compact. Therefore it is possible to choose open sets U^+ and U^- with

$$U^+ \supseteq Z^+, \quad U^- \supseteq Z^-,$$

$| p(x)| \leqslant 2 - \epsilon$ for all $p \in \mathrm{sp}\, P(f)$ with $\| p \| \leqslant 2$ and all $x \in U^+ \cup U^-$,

$$f(x) > 1 - \epsilon \qquad \text{for all} \quad x \in U^+,$$

and

$$f(x) < -1 + \epsilon \qquad \text{for all} \quad x \in U^-.$$

Now let V^+, V^- be closed sets with

$$Z^+ \subseteq \mathrm{int}\, V^+, \qquad V^+ \subseteq U^+,$$
$$Z^- \subseteq \mathrm{int}\, V^-, \qquad V^- \subseteq U^-.$$

Let $f_\epsilon \in C(X)$ be such that

$$f_\epsilon(x) = 1 \quad \text{for} \quad x \in V^+,$$
$$f_\epsilon(x) = -1 \quad \text{for} \quad x \in V^-,$$
$$f_\epsilon(x) = f(x) \quad \text{for} \quad x \notin U^+ \cup U^-,$$
$$f(x) \leqslant f_\epsilon(x) \leqslant 1 \quad \text{for} \quad x \in U^+$$

and

$$-1 \leqslant f_\epsilon(x) \leqslant f(x) \quad \text{for} \quad x \in U^-.$$

(The construction of such a function f_ϵ is, of course, similar to the construction of f' in Lemma 2.4.) Then (i) is satisfied. By Lemma 2.3, $0 \in P(f_\epsilon)$ and, if $p \in P(f_\epsilon)$, then $p \in \operatorname{sp} P(f)$ and also $\| p \| \leqslant 2$. Therefore,

$$f(x) - p(x) \geqslant (1 - \epsilon) - (2 - \epsilon) = -1 \quad \text{for all} \quad x \in U^+$$

while

$$f(x) - p(x) \leqslant f_\epsilon(x) - p(x) \leqslant 1 \quad \text{for all} \quad x \in U^+.$$

Consequently, $\| f - p \|_{U^+} \leqslant 1$. Similarly, $\| f - p \|_{U^-} \leqslant 1$. But

$$| f(x) - p(x) | = | f_\epsilon(x) - p(x) | \leqslant 1$$

for all $x \notin U^+ \cup U^-$. It follows that $\| f - p \| \leqslant 1$ and so $p \in P(f)$. This establishes (ii). Condition (iii) follows from the fact that if $p \in P(f_\epsilon)$, then $\{x : p(x) \geqslant 0\} \supseteq V^+$ and $\{x : p(x) \leqslant 0\} \supseteq V^-$.

Now suppose that $0 \neq q \in P(f)$. Let $f_0 = f - q$. Then $P(f_0) = P(f) - q$, and so $0 \in P(f_0)$. Also $\| f_0 \| = 1$. The lemma now follows by applying the special case that has been established to f_0, using the fact that

$$f_0^{-1}(1) \cap Z(P(f)) = f^{-1}(1) \cap Z(P(f)).$$
$$f_0^{-1}(-1) \cap Z(P(f)) = f^{-1}(-1) \cap Z(P(f)).$$

The next lemma is the basic result of this section.

LEMMA 2.6. *If* $0 \in P(f)$, $\| f \| = 1$ *and* $h \in P^*(f)$ *then, for every* $p \in P(f)$

$$\{x : h(x) \geqslant p(x)\} \textit{ is a neighbourhood of } f^{-1}(1) \cap Z(P(f))$$

and

$$\{x : h(x) \leqslant p(x)\} \text{ is a neighbourhood of } f^{-1}(-1) \cap Z(P(f)).$$

Proof. Suppose on the contrary that there is a $p_0 \in P(f)$ such that if U is any neighbourhood of $f^{-1}(1) \cap Z(P(f))$, then $h(x) < p_0(x)$ for some $x \in U$. Let $P = P(f)$ and let $q \in P(f)$, $r > 0$ be as in Lemma 2.1. Now for each $\epsilon > 0$ let f_ϵ have the properties (i), (ii) and (iii) of Lemma 2.5. Then it follows that $d(h, P(f_\epsilon)) \geqslant r$. Therefore, $h \notin P^*(f)$ which is a contradiction. The second conclusion of the lemma is proved similarly.

Results concerning lower semicontinuity of and continuous selections for P now follow easily. There are two theorems of which the first is given in the paper of Blatter, Morris and Wulbert.

THEOREM 2.7. [1]. *Let X be a compact Hausdorff space. Let M be a linear subspace of $C(X)$, P the metric projection onto M, and let M be such that $P(f)$ is nonempty and finite-dimensional for every $f \in C(X)$. In order that P be lower semicontinuous it is necessary and, if P is upper semicontinuous, it is also sufficient that $Z(P(f))$ be open for every $f \in C(X)$ with $0 \in P(f)$.*

Proof of Necessity. Suppose that P is lower semicontinuous, $f \in C(X)$ and $0 \in P(f)$. Let $w \in Z(P(f))$. It will be shown that $Z(P(f))$ is a neighbourhood of w.

Replacing f by $(1/\|f\|)f$ it may be supposed that $\|f\| = 1$. If $f(w) = -1$, let f be replaced by $-f$. If $|f(w)| < 1$, let f be replaced by the function f' of Lemma 2.4. Thus it may be supposed that $f(w) = 1$.

Now, by Proposition 1.1, $P^*(f) = P(f)$. Consider $p \in P(f)$. Then $0 \in P^*(f)$ and so, by Lemma 2.6, $\{x : p(x) \leqslant 0\}$ is a neighbourhood of w. However, $p \in P^*(f)$, and $0 \in P(f)$, and so, by Lemma 2.6 again, $\{x : p(x) \geqslant 0\}$ is also a neighbourhood of w. Thus $p^{-1}(0)$ is a neighbourhood of w. This is so for each $p \in P(f)$ which is a finite-dimensional set. It follows that $Z(P(f))$ is a neighbourhood of w.

Proof of Sufficiency. The proof of the sufficiency condition which is given in [1] is a little obscure. Therefore, for completeness we give here a simpler proof which originated in the present author's unpublished proof of the special case of the theorem in which X is a finite set [cf. 3, Theorem 3].

Suppose that $Z(P(f))$ is open whenever $0 \in P(f)$ and that P is upper semicontinuous. It is sufficient to show that $P^*(f) = P(f)$

for all $f \in C(X)$ with $0 \in P(f)$. Write $Z = Z(P(f))$. Let $p_0 \in \operatorname{relint} P(f)$. Then, by Lemma 2.2 applied to $f - p_0$,

$$|f(x) - p_0(x)| < d(f, M)$$

for all $x \notin Z$. But Z is open and X is compact and Hausdorff. Therefore

$$\|f - p_0\|_{X \setminus Z} < d(f, M).$$

Let $0 < \epsilon < \frac{1}{3}(d(f, M) - \|f - p_0\|_{X\setminus Z})$. By the upper semicontinuity of P there is a $\delta > 0$ such that $d(p', P(f)) < \epsilon$ for all $p' \in P(f')$ whenever $\|f - f'\| < \delta$. Suppose that $\|f - f'\| < \min\{\epsilon, \delta\}$ and $p' \in P(f')$. Then there exists $p \in P(f)$ such that $\|p' - p\| < \epsilon$. Now

$$\|f' - (p_0 + p' - p)\|_Z = \|f' - p'\|_Z \leqslant d(f', M),$$

and, since $d(f, M) \leqslant d(f', M) + \|f' - f\|$,

$$\begin{aligned}
&\|f' - (p_0 + p' - p)\|_{X\setminus Z} \\
&\quad \leqslant \|f' - f\|_{X\setminus Z} + \|f - p_0\|_{X\setminus Z} + \|p' - p\|_{X\setminus Z} \\
&\quad \leqslant 2\|f' - f\| + \|p' - p\| + d(f', M) - (d(f, M) - \|f - p_0\|_{X\setminus Z}) \\
&\quad \leqslant d(f', M).
\end{aligned}$$

Consequently, $p_0 + p' - p \in P(f')$ and so

$$d(p_0, P(f')) \leqslant \|p' - p\| < \epsilon.$$

It now follows that $p_0 \in P^*(f)$. This is so for every $p_0 \in \operatorname{relint} P(f)$. However $P^*(f)$ is closed and $\operatorname{relint} P(f)$ is dense in $P(f)$. The result now follows from Proposition 1.1.

A subspace M of $C(X)$ is said [2] to be a *Z-subspace* of $C(X)$ if the interior of $f^{-1}(0)$ is empty for every $f \in M \setminus \{0\}$. For Z-subspaces M we have the following theorem.

THEOREM 2.8. *Let X, M and P be as in Theorem 2.7 and let M also be a Z-subspace of $C(X)$. Then*

(i) *For each $f \in C(X)$ the set $P^*(f)$ contains at most one point,*

(ii) *If $P^*(f)$ is nonempty for every f in $C(X)$ then, whenever $0 \in P(f) \neq \{0\}$, $|f(x)| = \|f\|$ for all $x \in Z(P(f))$, and*

(iii) *Either there is no continuous selection for P or there is a unique one.*

Proof. (iii) is an immediate consequence of (i) and Proposition 1.1. In proving (i) and (ii) one may assume that $0 \in P(f)$ and $\|f\| = 1$.

Let $Z^+ = f^{-1}(1) \cap Z(P(f))$ and $Z^- = f^{-1}(-1) \cap Z(P(f))$. By Lemma 2.2, the union $Z^+ \cup Z^-$ is nonempty. It follows from Lemma 2.6 that if $h, h' \in P^*(f)$ then $\{x : h(x) = h'(x)\}$ is a neighbourhood of $Z^+ \cup Z^-$. But M is a Z-subspace of $C(X)$, and so $h = h'$. This proves (i).

To prove (ii) suppose that $0 \in P(f) \neq \{0\}$, $\|f\| = 1$ but that $|f(w)| < 1$ for some $w \in Z(P(f))$. Let f', f'' be the functions constructed in Lemma 2.4. Then by Lemma 2.6,

$$P^*(f') = P^*(f'') = P^*(f).$$

However, if $P^*(f) = \{h\}$, then, for any $p \in P(f)$, it follows, from Lemma 2.6 applied to f', that $\{x : h(x) \geqslant p(x)\}$ is a neighbourhood of w and, from Lemma 2.6 applied to f'', that $\{x : h(x) \leqslant p(x)\}$ is a neighbourhood of w. It now follows (using the finite-dimensionality of $P(f)$ as in the proof of Theorem 2.7) that $Z(P(f))$ is a neighbourhood of w which is not possible because M is a Z-subspace and $P(f) \neq \{0\}$.

3. A Non-Chebyshev Subspace with a Unique Continuous Selection

This section is devoted to establishing the existence of a Z-subspace M of $C([-1, 1])$ which is not a Chebyshev subspace but whose metric projection has a continuous selection. The example is of a five-dimensional subspace which contains the constant functions; it shows that two of the statements of [2] (Theorem 2.1 and Corollary 2.5) are false. There is an error in [2] on p. 205. In the course of the argument there, there is a "dimension reducing" construction which, it is claimed, replaces f with $\dim P(f) > 1$ by an f' with $\dim P(f') < \dim P(f)$. The argument at this point is not valid. However, the idea is relevant to the problem. In [2] it is said that a subspace M of $C(X)$ *changes sign* if for each f in M which takes both positive and negative values there is a point of X in each neighbourhood of which f takes both positive and negative values. Obviously, if X is connected then every Z-subspace M of $C(X)$ changes sign. Now if M contains a function which is everywhere positive, then for each $f \in C(X)$ with $0 \in P(f)$ the subsets Z^+ and Z^- of Lemma 2.5 are both nonempty. It then follows from Lemma 2.6 that, if M changes sign and $P(f)$ is one-dimensional, then $P^*(f)$ is empty. It is possible, by developing the dimension reducing argument of [2], to show that if (in the same circumstances) there exists an f such that $P(f)$ is two-dimensional, then there exists an f' such that $P^*(f')$ is empty. Thus in constructing an example of a non-Chebyshev subspace M with a continuous selection for the metric projection one must find an M such that for all $f \in C(X)$ either $\dim P(f) = 0$ or $\dim P(f) \geqslant 3$. If M is to contain an everywhere nonzero function, then M must have dimension at least four. However, in the example which will be constructed the subspace M will correspond, under the identification of 1 and -1, to a Chebyshev subspace of the space of continuous functions on a circle (and this feature is a consequence of the way in which we violate the condition of Haar's theorem for a subspace to be Chebyshev). Now a Chebyshev subspace on a circle is of odd dimension (see, e.g., [4, p. 26]) and so we are led to search for an example of dimension five. There is one other feature of the construction to be noted. Subspaces which are Chebyshev enter into the construction, and, therefore, by Mairhuber's theorem [5], the example is essentially an example on an interval or a circle.

The following terminology will be used. If $A \subseteq X$ and $f_0, f_1, ..., f_k \in C(X)$, then it will be said that $(f_0, f_1, ..., f_k)$ is a *Chebyshev system on* A if each function $f = \alpha_0 f_0 + \cdots + \alpha_k f_k$ with $|\alpha_0| + \cdots + |\alpha_k| > 0$ has at most k distinct zeros on A. It then follows from Haar's theorem that if B is a compact subset of A then the restrictions of the functions $f_0, ..., f_k$ to B span a $(k+1)$-dimensional Chebyshev subspace of $C(B)$. The theorem which follows provides the basis for the construction of our example.

THEOREM 3.1. *Let* $M = \mathrm{sp}\{h_0, h_1, h_2, h_3, h_4\} \subseteq C([-1, 1])$, *and let the following conditions be satisfied*:

(1) $h_0(t) = 1$ *for all* $t \in [-1, 1]$.

(2) $h_j(-1) = h_j(1) = 0$ for $j = 1, 2, 3, 4$.

(3) *For some* $\delta > 0$

$$h_4(t) > 0 \quad \textit{for all} \quad t \in (-1, -1 + \delta),$$

$$h_4(t) < 0 \quad \textit{for all} \quad t \in (1 - \delta, 1).$$

(4) $\lim_{t\to-1} h_j(t)/h_{j+1}(t) = \lim_{t\to1} h_j(t)/h_{j+1}(t) = 0$ *for* $j = 1, 2, 3$.

(5) (h_1), (h_1, h_2), (h_1, h_2, h_3) and (h_1, h_2, h_3, h_4) *are all Chebyshev systems on* $(-1, 1)$ *and* $(h_0, h_1, h_2, h_3, h_4)$ *is a Chebyshev system on* $(-1, 1]$.

Then there exists a unique continuous selection for P_M.

The five conditions under (5) will be referred to as (5) (i)–(v). The proof will be attained through a sequence of lemmas. In Lemmas 3.2–3.6 the function f in $C(X)$ will have dim $P(f) > 0$ and $0 \in$ relint $P(f)$.

LEMMA 3.2. (a) $|f(-1)| = |f(1)| = \|f\|$ *and* $1, -1 \in Z(P(f))$. (b) $P(f) \subseteq \mathrm{sp}\{h_1, h_2, h_3, h_4\}$.

Proof. Note that conditions (1), (2), and (5)(v) ensure that $(h_0, ..., h_4)$ is also a Chebyshev system on $[-1, 1)$.

Suppose, contrary to the lemma, that $|f(1)| < \|f\|$. Let $\epsilon = \frac{1}{2}(\|f\| - |f(1)|)$. Let $\lambda > 0$ be such that $|f(t)| < \|f\| - \epsilon$ for all $t \in (\lambda, 1]$. Then if $\|p\| < \epsilon$ we have $\|f - p\|_{[\lambda,1]} < \|f\|$. Suppose now that $q, -q \in P(f)$, and $0 < \|q\| < \epsilon$ (such q exist by the assumtions on $P(f)$). Then

$$\|f - q\|_{[-1,\lambda]} = \|f + q\|_{[-1,\lambda]} = \|f\| .$$

It now follows by the Chebyshevity of M on $[-1, \lambda]$ that there exists $p \in M$ such that $\|f - p\|_{[-1,\lambda]} < \|f\|$. It then follows that $\|f - \theta p\|_{[-1,\lambda]} < \|f\|$ for all $\theta \in (0, 1]$. Therefore, if $\theta \|p\| < \epsilon$, we have

$$\|f - \theta p\| = \max\{\|f - \theta p\|_{[\lambda,1]}, \|f - \theta p\|_{[-1,\lambda]}\} < \|f\| ,$$

which contradicts the fact that $0 \in P(f)$. This proves that $|f(1)| = \|f\|$. Similarly, $|f(-1)| = \|f\|$. That $1, -1 \in Z(P(f))$ now follows from Lemma 2.2. This proves (a), and (b) follows.

LEMMA 3.3. *If $p, q \in M$ are such that*

(i) $q, -q \in P(f)$

(ii) *there exists $\mu > 0$ such that p/q is bounded in*

$$A = (-1, -1 + \mu] \cup [1 - \mu, 1), \textit{ and}$$

(iii) *for each $\mu > 0$ there exists $\theta_0 > 0$ such that if $0 < \theta \leqslant \theta_0$*

$$\|f - \theta p\|_B < \|f\| \textit{ where } B = [-1 + \mu, 1 - \mu],$$

then there exists θ_1 such that $\theta p \in P(f)$ for all $\theta \in (0, \theta_1]$.

Proof. Let μ be as in condition (ii) and let θ_0 correspond to μ as in (iii). Then, by (ii) there exists $\theta_1 \leqslant \theta_0$ such that $|\theta_1 p(t)| \leqslant |q(t)|$ for all $t \in A$. Now

$$\|f - q\| = \|f + q\| = \|f\|$$

and, therefore,

$$-\|f\| + |q(t)| \leqslant f(t) \leqslant \|f\| - |q(t)|$$

for all $t \in [-1, 1]$. Consequently, if $0 < \theta < \theta_1$, then

$$\|f - \theta p\|_A \leqslant \|f\|.$$

The conclusion of the lemma now follows.

LEMMA 3.4. *If* $t \in (-1, 1)$ *then* $|f(t)| < \|f\|$.

Proof. By Lemma 3.2(b) and the assumptions on $P(f)$ there is $q \in P(f)$ such that $-q \in P(f)$ and $q = \alpha_1 h_1 + \cdots + \alpha_k h_k$ with $\alpha_k > 0$ where k is one of 1, 2, 3 and 4.

Suppose, contrary to the lemma, that $|f(t_0)| = \|f\|$ for some $t_0 \in (-1, 1)$. By Lemma 2.2, $t_0 \in Z(P(f))$ and therefore

$$|f(t_0) - p(t_0)| = \|f\| \qquad \text{for all} \quad p \in P(f).$$

We shall arrive at a contradiction to this last statement.

By condition (5)(v), there are at most 3 points of $Z(P(f))$ in $(-1, 1)$. Choose $\lambda > 0$ so that $(-1, -1 + \lambda] \cup [1 - \lambda, 1)$ contains no point of $Z(P(f))$. Let $I = [-1 + \lambda, 1 - \lambda]$. Then $t_0 \in I$ and $\|f - q\|_I = \|f\|_I$. Then by the Chebyshevity of $(h_1, \ldots, h_k)$ on $(-1, 1)$ [conditions 5(i)–(iv)] there exists $p \in \mathrm{sp}\{h_1, \ldots, h_k\}$ such that $\|f - p\|_I < \|f\|$. Then for $0 < \theta \leqslant 1$ we have $\|f - \theta p\|_I < \|f\|$. Now by condition (4) the functions p, q satisfy condition (ii) of Lemma 3.3. Suppose that $0 < \mu$ and also that $\mu < \lambda$. Let

$$J = [-1 + \mu, -1 + \lambda] \cup [1 - \lambda, 1 - \mu].$$

By Lemma 2.2 and the choice of λ we have $\|f\|_J < \|f\|$ and so $\|f - \theta p\|_J < \|f\|$ if $\theta \|p\| < (\|f\| - \|f\|_J)$. It now follows that p satisfies condition (iii) of Lemma 3.3. Thus if θ is small $\theta p \in P(f)$ while $|f(t_0) - \theta p(t_0)| \leqslant \|f - \theta p\|_I < \|f\|$ which is the required contradiction.

LEMMA 3.5. $f(-1)$ *and* $f(1)$ *have opposite sign.*

Proof. If this were not so, then, by Lemma 3.4 and the compactness of $[-1, 1]$, we would have $\|f - \theta h_0\| < \|f\|$ for some θ which would contradict the assumption that $0 \in P(f)$.

LEMMA 3.6. $\mathrm{sp}\, P(f) = \mathrm{sp}\{h_1, h_2, h_3, h_4\}$.

Proof. It may be supposed, by Lemma 3.5, that $f(-1) = -f(1) = \|f\|$. It now follows from Lemma 3.4 and condition (3) of the theorem that if $\theta > 0$ is small, then $\|f - \theta h_4\| \leqslant \|f\|$ and so $\theta h_4 \in P(f)$. But $0 \in \operatorname{relint} P(f)$ and so, for some $\theta_0 > 0$, both $q = \theta_0 h_4$ and $-\theta_0 h_4$ are in $P(f)$. Now taking $p = h_j$ ($j = 1, 2$ and 3) it follows from Lemma 3.3 (using Lemma 3.4 and condition (4) of the theorem) that $P(f)$ contains nonzero multiples of h_1, h_2 and h_3.

LEMMA 3.7. *If $p \in \operatorname{relint} P(f)$ and $\dim P(f) > 0$ then*

$$|f(t) - p(t)| = \|f - p\|$$

if and only if $t = 1$ or $t = -1$.

Proof. The lemma follows from Lemmas 3.2(a) and 3.4 applied to the function $f - p$.

Let α be the continuous linear functional defined on M by

$$\alpha(\alpha_0 h_0 + \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3 + \alpha_4 h_4) = \alpha_4 .$$

LEMMA 3.8. *If $f \in C(X)$ has $\dim P(f) > 0$ and $f(-1) - q(-1) = \|f - q\|$ when $q \in P(f)$ then there is a unique $p \in P(f)$ such that*

$$\alpha(p) = \sup\{\alpha(q) : q \in P(f)\}.$$

Proof. It follows from Lemma 3.2(a) that the condition $f(-1) - q(-1) = \|f - q\|$ is independent of $q \in P(f)$. We can suppose that $0 \in \operatorname{relint} P(f)$ and, by Lemmas 3.2 and 3.5, that $f(-1) = -f(1) = \|f\|$.

The set $P(f)$ is compact and α is continuous and so there is at least one $p_0 \in P(f)$ such that $\alpha(p_0) = \sup\{\alpha(q) : q \in P(f)\}$. We will now write $\alpha(p_0) = \alpha_m$. By Lemma 3.6, since $0 \in \operatorname{relint} P(f)$, we have $\alpha_m > 0$.

Let $I = [-1 + \delta, 1 - \delta]$ where δ is that of condition (3) in the theorem. Then $\| \cdot \|_I$ is a norm on the subspace $\operatorname{sp}\{h_1, h_2, h_3\}$ of $C([-1, 1])$ equivalent to any other norm. It therefore follows from condition (4) of the theorem that there exists λ with $0 < \lambda < \delta$ such that

$$|q(t)| \leqslant \alpha_m |h_4(t)| \tag{6}$$

for all $t \in [-1, -1 + \lambda] \cup [1 - \lambda, 1]$ and all those $q \in \operatorname{sp}\{h_1, h_2, h_3\}$ which have $\|q\|_I \leqslant \|f\| + \|f - \alpha_m h_4\|$.

The set $f - P(f)$ is equicontinuous at 1 and -1 and so we may also suppose that for every $p \in P(f)$

$$\begin{aligned} f(t) - p(t) &\geqslant 0 \quad \text{for all} \quad t \in [-1, -1 + \lambda], \\ f(t) - p(t) &\leqslant 0 \quad \text{for all} \quad t \in [1 - \lambda, 1]. \end{aligned} \tag{7}$$

Let $J = [-1 + \lambda, 1 - \lambda]$. Suppose that $p \in P(f)$ and that $\alpha(p) = \alpha_m$. Then

$$\| f - p \|_J = \| f \| . \tag{8}$$

For if, on the contrary, $\| f - p \|_J < \| f \|$ then by (7) and condition (3) we would have (since $\lambda < \delta$) $\| f - (p + \theta h_4) \| = \| f \|$ for some (small) $\theta > 0$, which would contradict the maximality of $\alpha(p) = \alpha_m$.

Now suppose that $p_1 \in P(f)$, $p_1 \neq p_0$ but that $\alpha(p_1) = \alpha_m$. We can write $p_0 = q_0 + \alpha_m h_4$, $p_1 = q_1 + \alpha_m h_4$ where $q_0, q_1 \in \operatorname{sp}\{h_1, h_2, h_3\}$. Then, by (8),

$$\|(f - \alpha_m h_4) - q_0 \|_J = \|(f - \alpha_m h_4) - q_1 \|_J = \| f \| .$$

It now follows from the Chebyshevity of (h_1, h_2, h_3) on J (condition 5(iii)) that there exists $q \in \operatorname{sp}\{h_1, h_2, h_3\}$ with

$$\| f - \alpha_m h_4 - q \|_J < \| f \|.$$

Let $q_\theta = (1 - \theta)\, q_0 + \theta q$. Then

$$f - \alpha_m h_4 - q_\theta = f - p_0 + \theta(q_0 - q).$$

If $0 < \theta \leqslant 1$ then

$$\| f - \alpha_m h_4 - q_\theta \|_J < \| f \| \tag{9}$$

In this case $q_\theta \in \operatorname{sp}\{h_1, h_2, h_3\}$ and

$$\| q_\theta \|_I \leqslant \| q_\theta \|_J < \| f - \alpha_m h_4 \| + \| f \|.$$

Therefore inequality (6) holds for q_θ and so

$$\begin{aligned} f(t) - \alpha_m h_4(t) - q_\theta(t) &\leqslant f(t) \leqslant \| f \| \quad \text{for all} \quad t \in [-1, -1 + \lambda], \\ f(t) - \alpha_m h_4(t) - q_\theta(t) &\geqslant f(t) \geqslant - \| f \| \quad \text{for all} \quad t \in [1 - \lambda, 1]. \end{aligned}$$

However, by (7), if $\theta \| q - q_0 \| \leqslant \| f \|$, then

$$\begin{aligned} f(t) - p_0(t) + \theta(q_0(t) - q(t)) &\geqslant - \| f \| \quad \text{for all} \quad t \in [-1, -1 + \lambda], \\ f(t) - p_0(t) + \theta(q_0(t) - q(t)) &\leqslant \| f \| \quad \text{for all} \quad t \in [1 - \lambda, 1]. \end{aligned}$$

Thus, if $0 < \theta \leqslant 1$ and $\theta \| q - q_0 \| \leqslant \| f \|$, then

$$\| f - \alpha_m h_4 - q_\theta \| \leqslant \| f \| ,$$

so that $\alpha_m h_4 + q_\theta \in P(f)$ and $\alpha(\alpha_m h_4 + q_\theta) = \alpha_m$. But then (9) contradicts (8).

It is now possible to define a selection for P_M in the following way. If $\dim P(f) = 0$ let $s(f)$ be the unique member of $P(f)$. If $\dim P(f) > 0$ and, for $q \in P(f)$, $f(-1) - q(-1) = \| f - q \|$, then let $s(f)$ be (unique by Lemma 3.8) $p \in P(f)$ such that

$$\alpha(p) = \sup\{\alpha(q) : q \in P(f)\}.$$

If $\dim P(f) > 0$ and, for $q \in P(f)$, $f(-1) - q(-1) = - \| f - q \|$ let $s(f) = -s(-f)$, $s(-f)$ having been defined. By Lemma 3.7, $s(f)$ is defined for all $f \in C(X)$. Furthermore, for any λ and any $p \in M$, $s(\lambda f + p) = \lambda s(f) + p$. The next lemma shows that s is a continuous selection for P_M. That it is the unique one is then a consequence of Theorem 2.8.

LEMMA 3.9. *s is continuous.*

Proof. It must be shown that s is continuous at each $f \in C([-1, 1])$. If $\dim P(f) = 0$, then s is continuous at f by the upper semicontinuity of P (and the fact that s is a selection). Suppose that $\dim P(f) > 0$ and $q \in \operatorname{relint} P(f)$. Then s is continuous at f if and only if s is continuous at $f - q$ and also if and only if s is continuous at $-(f - q)$. Therefore, we may assume that $0 \in \operatorname{relint} P(f)$ and that $f(-1) = \| f \|$.

Suppose that s is not continuous at f. Then there is a sequence $(f_n)_{n \geqslant 1}$ in $C([-1, 1])$ and $p \in P(f)$ such that $f = \lim f_n$, $p = \lim s(f_n)$ and $p \neq s(f)$. By Lemma 3.8, $\alpha(p) < \alpha(s(f))$. Let $q \in M$ be such that $p + q \in \operatorname{relint} P(f)$ and $\alpha(p + q) > \alpha(p)$. Then by Lemma 3.4

$$| f(t) - (p(t) + q(t)) | < \| f \| = \| f - p \| \tag{10}$$

for all $t \in (-1, 1)$. By conditions (3) and (4) of the theorem and the fact that $f(-1) = \| f \| > 0$, there exists $\lambda > 0$ such that

$$f(t) - p(t) - q(t) \geqslant \tfrac{1}{2} \| f \| \qquad \text{and} \quad q(t) \geqslant 0,$$

for all $t \in [-1, -1 + \lambda]$,

$$f(t) - p(t) - q(t) \leqslant -\tfrac{1}{2} \| f \| , \qquad \text{and} \quad q(t) \leqslant 0$$

for all $t \in [1 - \lambda, 1]$. Let $J = [-1 + \lambda, 1 - \lambda]$. Then by (10) $\|f - p - q\|_J < \|f - p\|$. It now follows that if n is sufficiently large (so that $\|f - f_n\|$ and $\|p - s(f_n)\|$ are small then

$$f_n(t) - s(f_n)(t) \geqslant f_n(t) - s(f_n)(t) - q(t) \geqslant 0 \quad \text{for all} \quad t \in [-1, -1 + \lambda],$$

$$f_n(t) - s(f_n((t) \leqslant f_n(t) - s(f_n)(t) - q(t) \leqslant 0 \quad \text{for all} \quad t \in [1 - \lambda, 1],$$

and

$$\|f_n - s(f_n) - q\|_J < \|f_n - s(f_n)\| .$$

From these inequalities it follows that

$$\|f_n - s(f_n) - q\| \leqslant \|f_n - s(f_n)\|$$

and so $s(f_n) + q \in P(f_n)$. Thus $\dim P(f_n) > 0$ and

$$f_n(-1) - s(f_n)(-1) > 0;$$

and therefore we have a contradiction to the definition of $s(f_n)$. This completes the proof of the lemma and of the theorem.

We can now prove the following theorem.

THEOREM 3.10. *There exists a five-dimensional Z-subspace of $C([-1, 1])$ which contains the constants, is non-Chebyshev and has a unique continuous selection for its metric projection.*

Proof. We must construct functions $h_0, \ldots, h_4$ which satisfy the conditions (1)–(5) of Theorem 3.1. Define functions $k_0, \ldots, k_4$ in $C([-1, 1])$ by $k_0(t) = 1$, $k_1(t) = t^4$, $k_2(t) = t^3(1 - t^2)$, $k_3(t) = t^2$ and $k_4(t) = t(1 - t^2)$. Now define

$$h_j(t) = \begin{cases} k_j(t + 1) & \text{for} \quad -1 \leqslant t \leqslant 0 \\ k_j(t - 1) & \text{for} \quad 0 \leqslant t \leqslant 1 \end{cases}$$

for $j = 0, 1, \ldots, 4$. Then the functions $h_j (0 \leqslant j \leqslant 4)$ are in $C([-1, 1])$ and conditions (1)–(5) are satisfied. That conditions (1)–(4) are satisfied is immediate. To show that (5) is satisfied we make the following observation: If p is a polynomial of degree $\leqslant r$, where r is odd and $p(-1) = p(1)$ then p has at most $r - 1$ zeros in $(-1, 1]$. Now consider e.g., a polynomial p in $\operatorname{sp}\{k_1, k_2\}$:

$$p(t) = \alpha_1 k_1(t) + \alpha_2 k_2(t) = t^2(\alpha_1 t^2 + \alpha_2 t(1 - t^2)).$$

By our observation the second factor on the right has at most two zeros in $(-1, 1]$ one of which is at 0. Therefore, p has at most one zero in

$(-1, 0) \cup (0, 1]$ and the corresponding function in $\mathrm{sp}\{h_1, h_2\}$ has at most one zero in $(-1, 1)$. This proves that (h_1, h_2) is a Chebyshev system on $(-1, 1)$. The other parts of (5) are verified similarly. The proof of the theorem is complete.

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